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FREE VIBRATIONS OF A STEPPED, SPINNING TIMOSHENKO BEAM

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1. INTRODUCTION

An original way of accurately calculating bending moments and shear forces for an Euler–Bernoulli beam having discontinuities has been outlined recently [1]. The approach uses generalized force mode functions obtained from a simple uniform beam having the same end conditions as the beam with discontinuities. Such functions may be constructed by finding the static deflection of the uniform beam arising from either a concentrated moment or force acting at the location of a discontinuity. However, a boundary value problem has to be disadvantageously solved. An alternative procedure used here avoids this difficulty by constructing the functions directly from polynomials. The procedure involves two steps. First, polynomials are found on each side of a discontinuity that satisfy the conditions at the contiguous end. Second, the polynomials must be chosen so that the transverse deflection and its slope (or the slope due to bending for a Timoshenko beam [2]) are continuous at the location of a discontinuity.

The objective of the present note is to employ polynomial based generalized force mode functions, with the method of Galerkin [3] rather than Rayleigh–Ritz, to solve an illustrative problem that, for the first time, is not self-adjoint. The problem involves the free vibrations of the simply supported but stepped spinning Timoshenko beam shown in Figure 1. The analogous uniform beam, however, uses the Euler–Bernoulli simplification and assumes no spinning.

2. OUTLINE OF ANALYSIS

Consider a Timoshenko beam having length L and a circular cross-section which is discontinuous at $x = L/2$. Suppose that the beam spins at a constant angular speed, Ω, about the x-axis which coincides with the beam's geometric centre in the fixed (inertial) co-ordinate frame of Figure 1. The beam has mass density, ρ, Young's modulus, E, shear modulus, G, and shear coefficient k. Let $A(x)$, $I(x)$ and J_x be the area, moment and polar moment of inertia of a cross-section that is distance x from the left end. The transverse deflections in the y and z directions are designated $u_y(x)$ and $u_z(x)$, respectively, whilst ψ_y and ψ_z represent the analogous slope due to bending. The free vibrations of the spinning beam are governed by [2]

$$-(kAG(\psi_y + \mathrm{d}u_y/\mathrm{d}x))' + \zeta^2\rho A u_y = 0, \qquad -(kAG(\psi_z + \mathrm{d}u_z/\mathrm{d}x))' + \zeta^2\rho A u_z = 0, \quad (1, 2)$$

$$-(EI\,\mathrm{d}\psi_y/\mathrm{d}x)' + kAG(\psi_y + \mathrm{d}u_y/\mathrm{d}x) + \zeta^2\rho I\psi_y + \zeta\Omega\rho J_x\psi_z = 0 \quad (3)$$

and

$$-(EI\,\mathrm{d}\psi_z/\mathrm{d}x)' + kAG(\psi_z + \mathrm{d}u_z/\mathrm{d}x) + \zeta^2\rho I\psi_z - \zeta\Omega\rho J_x\psi_y = 0, \quad (4)$$

where a prime superscript indicates differentiation with respect to x whilst $\zeta = \omega\mathrm{i}$,

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$\mathrm{i} = (-1)^{1/2}$ and ω is a natural frequency. The simply supported ends are denoted by

$$u_y(0) = u_y(L) = u_z(0) = u_z(L) \tag{5}$$

and

$$\psi_y'(0) = \psi_y'(L) = \psi_z'(0) = \psi_z'(L) = 0. \tag{6}$$

On the other hand, the force compatibility conditions for the discontinuity at $x = L/2 \equiv x_0$ are

$$\frac{\mathrm{d}\psi_y(x_0^-)}{\mathrm{d}x} = \frac{EI(x_0^+)}{EI(x_0^-)}\frac{\mathrm{d}\psi_y(x_0^+)}{\mathrm{d}x}, \qquad \frac{\mathrm{d}\psi_z(x_0^-)}{\mathrm{d}x} = \frac{EI(x_0^+)}{EI(x_0^-)}\frac{\mathrm{d}\psi_z(x_0^+)}{\mathrm{d}x}, \tag{7}$$

$$\left(\psi_y(x_0^-) + \frac{\mathrm{d}u_y(x_0^-)}{\mathrm{d}x}\right) = \frac{kGA(x_0^+)}{kGA(x_0^-)}\left(\psi_y(x_0^+) + \frac{\mathrm{d}u_y(x_0^+)}{\mathrm{d}x}\right) \tag{8}$$

and

$$\left(\psi_z(x_0^-) + \frac{\mathrm{d}u_z(x_0^-)}{\mathrm{d}x}\right) = \frac{kGA(x_0^+)}{kGA(x_0^-)}\left(\psi_z(x_0^+) + \frac{\mathrm{d}u_z(x_0^+)}{\mathrm{d}x}\right). \tag{9}$$

Assume that approximate solutions have the form

$$u_y^n = \sum_{j=1}^{n} \delta_{1j}^n \phi_{1j}(x), \qquad u_z^n = \sum_{j=1}^{n} \delta_{2j}^n \phi_{1j}(x) \tag{10}$$

and

$$\psi_y^n = \sum_{j=1}^{n} \delta_{3j}^n \phi_{2j}(x), \qquad \psi_z^n = \sum_{j=1}^{n} \delta_{4j}^n \phi_{2j}(x). \tag{11}$$

The $\phi_{ij}(x)$ ($i = 1, 2$ and $j = 1, 2, \ldots, n$) are admissible functions whilst the δ_{lj}^n ($l = 1, 2, 3, 4$) are undetermined coefficients. Substituting these forms into the left sides of equations (1)–(4) leads to the residual errors

$$\varepsilon_1^n = -\left(kAG\left(\psi_y^n + \frac{\mathrm{d}u_y^n}{\mathrm{d}x}\right)\right)' + \zeta^2\rho A u_y^n, \qquad \varepsilon_2^n = -\left(kAG\left(\psi_z^n + \frac{\mathrm{d}u_z^n}{\mathrm{d}x}\right)\right)' + \zeta^2\rho A u_z^n, \tag{12, 13}$$

$$\varepsilon_3^n = -\left(EI\frac{\mathrm{d}\psi_y^n}{\mathrm{d}x}\right)' + kAG\left(\psi_y^n + \frac{\mathrm{d}u_y^n}{\mathrm{d}x}\right) + \zeta^2\rho I\psi_y^n + \zeta\Omega\rho J_x\psi_z^n \tag{14}$$

$$\varepsilon_4^n = -\left(EI\frac{\mathrm{d}\psi_z^n}{\mathrm{d}x}\right)' + kAG\left(\psi_z^n + \frac{\mathrm{d}u_z^n}{\mathrm{d}x}\right) + \zeta^2\rho I\psi_z^n - \zeta\Omega\rho J_x\psi_y^n. \tag{15}$$

Coefficients δ_{lj}^n are determined from the requirements that [3]

$$\int_0^L \phi_{1j}(x)\varepsilon_1^n\,\mathrm{d}x = 0, \qquad \int_0^L \phi_{1j}(x)\varepsilon_2^n\,\mathrm{d}x = 0, \qquad \int_0^L \phi_{2j}(x)\varepsilon_3^n\,\mathrm{d}x = 0 \quad \text{and} \int_0^L \phi_{2j}(x)\varepsilon_4^n\,\mathrm{d}x = 0,$$

$$j = 1, 2, \ldots, n, \tag{16}$$

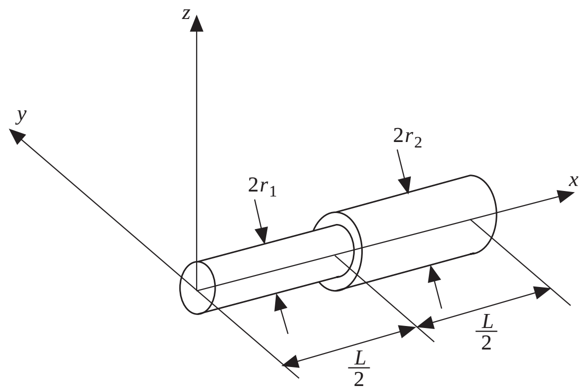

Figure 1. The inertial co-ordinates, xyz.

that lead, in matrix notation, to

$$(\zeta^2[\mathbf{M}] + \zeta[\mathbf{C}] + [\mathbf{K}])\{\delta^n\} = \{\mathbf{0}\}, \tag{17}$$

where

$$\{\delta^n\} = (\delta^n_{11} \cdots \delta^n_{1n}, \ldots, \delta^n_{41} \cdots \delta^n_{4n})^{\mathrm{T}}. \tag{18}$$

The [**M**] and [**K**] are the symmetric mass and stiffness matrix, respectively, whilst [**C**] is the skew-symmetric, gyroscopic matrix. Equation (17) represents a system that is not self-adjoint. It can be rewritten as

$$\begin{bmatrix} \mathbf{0} & [\mathbf{I}] \\ -[\mathbf{M}]^{-1}[\mathbf{K}] & -[\mathbf{M}]^{-1}[\mathbf{C}] \end{bmatrix} \begin{Bmatrix} \{\delta^n\} \\ \zeta\{\delta^n\} \end{Bmatrix} = \zeta \begin{Bmatrix} \{\delta^n\} \\ \zeta\{\delta^n\} \end{Bmatrix} \tag{19}$$

in order to employ a standard eigenvalue solver. A specific beam that has the material and dimensional properties given in Table 1 is considered next.

3. NUMERICAL RESULTS

The first and second order deflection derivatives as well as the slope due to bending of the example beam are discontinuous at its stepped midpoint, $x = 0{\cdot}5$ m. Consequently, the corresponding derivatives of the generalized force mode functions must also be discontinuous at this location. They should also satisfy the contiguous end conditions. When used, the generalized force mode functions are designated arbitrarily in equations (10) and (11) by $\phi_{11}(x)$ and $\phi_{12}(x)$ for the deflection, and by $\phi_{21}(x)$ and $\phi_{22}(x)$ for the slope due to bending. Their piecewise polynomial forms, obtained by following the procedure outlined in the introduction, are summarized in Table 2 for the situation when $L = 1$ m.

TABLE 1

Properties of the spinning beam

L (m)	r_1 (m)	r_2 (m)	ρ (kg/m^3)	k	E (GPa)	G (GPa)	Ω (rad/s)
1	0·01	0·012	7833·5	0·9	200	83	200

TABLE 2

Generalized force mode functions in the inertial co-ordinate frame

Generalized force mode functions for the deflection	Generalized force mode functions for the slope due to bending
$\phi_{11} = x, \quad \phi_{12} = x(4x^2 - 1) \quad 0{\cdot}0 \leqslant x \leqslant 0{\cdot}5$	$\phi_{21} = 4x^2 - 1, \quad \phi_{22} = 12x^2 - 1 \quad 0{\cdot}0 \leqslant x \leqslant 0{\cdot}5$
$\phi_{11} = 1 - x,$	$\phi_{21} = -4x^2 + 8x - 3,$
$\phi_{12} = 4x^3 - 12x^2 + 11x - 3 \quad 0{\cdot}5 \leqslant x \leqslant 1{\cdot}0$	$\phi_{22} = 12x^2 - 24x + 11 \quad 0{\cdot}5 \leqslant x \leqslant 1{\cdot}0$

TABLE 3

Values of ω for a stepped, simply supported, spinning beam

Mode number	Present method		No force mode functions		Exact results	
1	264·53	−264·41	266·04	−265·92	264·53	−264·41
2	1104·84	−1104·37	1105·23	−1104·76	1104·70	−1104·23
3	2415·35	−2414·27	2428·57	−2427·49	2415·32	−2414·25
4	4364·84	−4362·74	4367·21	−4365·38	4361·92	−4360·10

− indicates a backward precession frequency.

They possess the properties

$$\phi'_{12}(x) = \phi_{22}(x), \qquad \phi'_{21}(x) = 8\phi_{11}(x), \qquad x \neq 0{\cdot}5 \text{ m}, \tag{20}$$

and

$$\phi'_{22}(x) = \begin{cases} 24\phi_{11}(x), & x \leqslant 0{\cdot}5 \text{ m}, \\ -24\phi_{11}(x), & x \geqslant 0{\cdot}5 \text{ m}, \end{cases} \tag{21}$$

that make the calculation of [**K**] in equation (19) easier [5]. The remaining admissible functions are taken to be the eigenfunctions of a uniform, non-spinning Euler–Bernoulli beam having simply supported ends for the deflection and sliding–sliding ends for the slope due to the bending, i.e., $\phi_{1j}(x) = \sin(j-2)\pi x/L$ and $\phi_{2j} = \cos(j-3)\pi x/L$ for

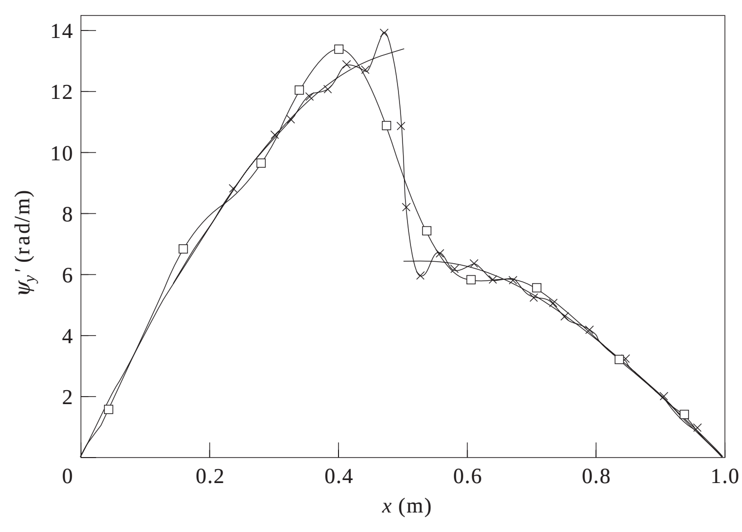

Figure 2. Exact and numerical values of ψ_y'. ——, Exact with GFMF† with $n = 10$ and $n = 35$; ⊟, $n = 10$, no GFMF; ⨯, $n = 35$, no GFMF. † Generalized force mode functions.

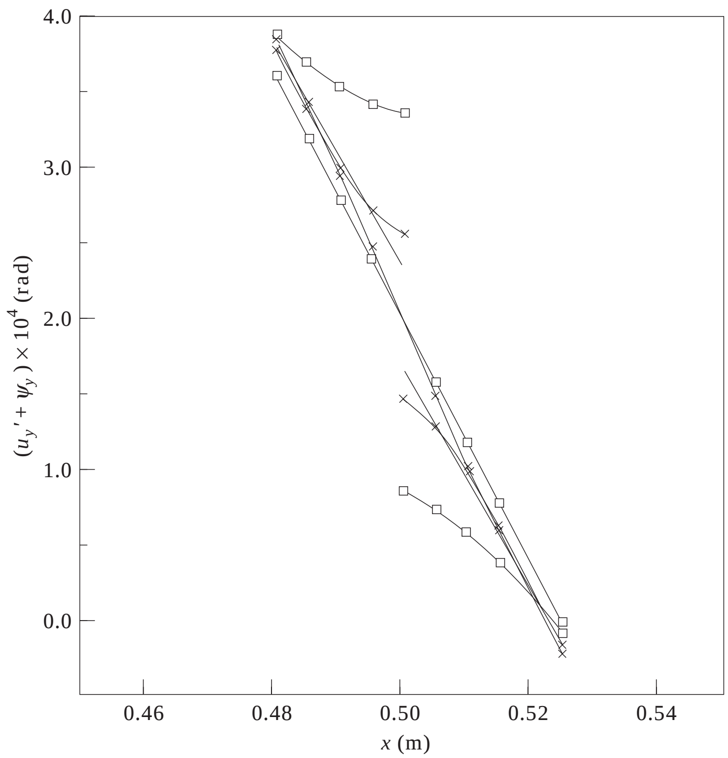

Figure 3. Exact and numerical values of $u_y' + \psi_y$. ——, Exact; —◇—, $n=10$, GFMF; +, $n = 35$, GFMF; —□—, $n = 10$, no GFMF; — × —, $n = 35$, no GFMF.

$j = 3, \ldots, n$. It can be shown [6] that all the $\phi_{1j}(x)$ and $\phi_{2j}(x)$, $j \geqslant 1$, satisfy the end conditions (5) and (6), respectively. The resulting numerical data for the first four forward and backward precession frequencies, computed with $n = 10$ in equations (10) and (11), are presented in Table 3 alongside the exact values. Data calculated without the generalized force mode functions are also given. Then $\phi_{1j}(x) = \sin j\pi x/L$ and $\phi_{2j} = \cos (j-1)\pi x/L$ are employed in equations (10) and (11) for $j = 1, \ldots, n$ where $n = 10$ again. It can be seen that the generalized force mode functions certainly improve the accuracy of the natural frequencies.

To ascertain if Gibbs phenomenon [1] occurs in the bending moment and shear force due to the stepped cross-section, the $\psi_y'(x)$ and $u_y'(x) + \psi_y(x)$ for the first forward precession frequency are compared with their exact values in Figures 2 and 3, respectively. Corresponding results that were computed without the generalized force mode functions, as before, are also presented. For convenience, $u_y(x)$ is invariably taken as 1 m at the beam's midpoint. Figure 2 demonstrates that the exact results and those obtained with the inclusion of the generalized force mode functions overlap, despite the discontinuous nature of the derivatives. However, the data obtained without these functions oscillate around the beam's midpoint. A similar oscillation can also be found in Figure 3. Furthermore, this last figure suggests that the numerical data obtained with the generalized force mode functions converge to the exact results with an increasing n.

4. CONCLUSION

Generalized force mode functions enable the free vibrations of a non-self-adjoint Timoshenko beam problem to be found without the Gibbs phenomenon occurring.

ACKNOWLEDGMENT

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